

Effective Characterization of semi-abelian varieties (by Jeff Baurdin)

§ Tate Philosophy

§ WWPB equivalence

§ semi-abelian varieties

§ Results (Sketch of argument)

Proj world

$\nabla \phi$

X smooth proj variety

Ω'_X - sheaf of 1-form

$n := \dim X$ rank n vb

$\wedge^n \Omega'_X =: \omega_X$
canonical sheaf

$q(X) := h^0(X, \Omega'_X)$

$P_m(X) := h^0(X, \omega_X^{\otimes m})$

$K(X)$ Kodaira
dim.

Quasi Projective world

V smooth quasi projective

$\hookrightarrow (X, D)$ X smooth
projective D snc divisor

$$V = X \setminus D$$

$\Omega'_X(\log D)$ meromorphic

1-forms on X which have
poles at most log on D

vb of rank $n = \dim V = \dim X$

$$\wedge^n \Omega'_X(\log D) \cong \omega_X(D)$$

$\overline{q}(V) := h^0(X, \Omega'_X(\log D))$

$\overline{P}_m(V) = h^0(X, \omega_X(D)^{\otimes m})$

$\overline{K}(V) = K(X, \omega_X(D))$

- The invariants s do not depend of X
- They are not birational invariant

Ex

A abelian variety

D smooth ample divisor

$A, D \subset \mathbb{P}^n$ are birationally
equivalent

but they have not the same
invariants

litaka Philosophy : to any statement
in the proj world which is governed
by forms there should correspond another
statement in the quasi projective
world governed by logarithmic form.

Today: The log analogue of

Thm (Chen-Hacon 2001)

X smooth projective X is birationally
equivalent to an abelian variety $\Leftrightarrow$

$$\underline{P_1(X) = P_2(X) = 1}$$

$$\underline{\dim(X) = g(X)}$$

Why is this a "good" statement for the LP

One proves that

$\text{alb}_X: X \longrightarrow \text{Alb}(X)$ is birational

↗ "Governed" by forms.

① Birationally equivalence is not good
what is the right equivalence
relation to consider?

② What is the analogue of an
Abelian variety?

§ WWPB equivalence (After Italeo)

Which birational maps preserve log. invariant

a. Proper birational maps are ok

b. $i: V \hookrightarrow W$ with codim of complement at least c .

Def we say that $\varphi: V \dashrightarrow W$ is a weak
Proper birational equivalence (WPB)
if it is a composition of a & b &
their birational inverses

$$\text{Bl}_p(S) \in \text{Bl}_p(S) \xrightarrow{\quad} \text{Bl}_p(S)$$

$\downarrow \text{WPB}$

$$S \setminus p \xrightarrow{\text{WPB}} \left(\begin{array}{c} \overline{} \\ \cdot p \\ \underline{} \end{array} \right) S$$

$$\mathcal{W} := \left\{ f: V \rightarrow W \mid \begin{array}{l} \text{either } f = g \text{ or } h \circ f \\ \text{for some } g \text{ or } h \text{ WPB is WPB} \end{array} \right\}$$

$\varphi: V \dashrightarrow W$ is a $\mathcal{W}$ WPB if it is
compositions of things in $\mathcal{W}$ & then bir inv.

you can go or

$\mathcal{W}^\infty \text{PB}$

this is the weakest
possible binationality
relation that preserves log inv.

We will check

$\mathcal{W} \text{WPB}$ equivalence.

§ Semi-Abelian varieties PROJ

A abelian variety

$$A \cong \mathbb{C}^g / \Lambda \quad \begin{array}{l} \text{rank } 2g \\ \text{discrete} \\ \text{subgroup} \end{array}$$

$$\mathcal{L}_A = \mathcal{O}_A^{\oplus g}$$

$$\omega_A = \mathcal{O}_A$$

$$q(A) = \dim A = g$$

$$P_m(A) = 1 \quad k(A) = 0$$

(Quasi-Abelian) Quasi PROJ.

G algebraic group
sits

$$0 \rightarrow G_m^r \rightarrow G \rightarrow A \rightarrow 0$$

abelian variety

$\exists$ 2 compactification

$$(Z, \Delta)$$

$$\mathcal{L}_Z(\log \Delta) = \mathcal{O}_Z^{\dim Z}$$

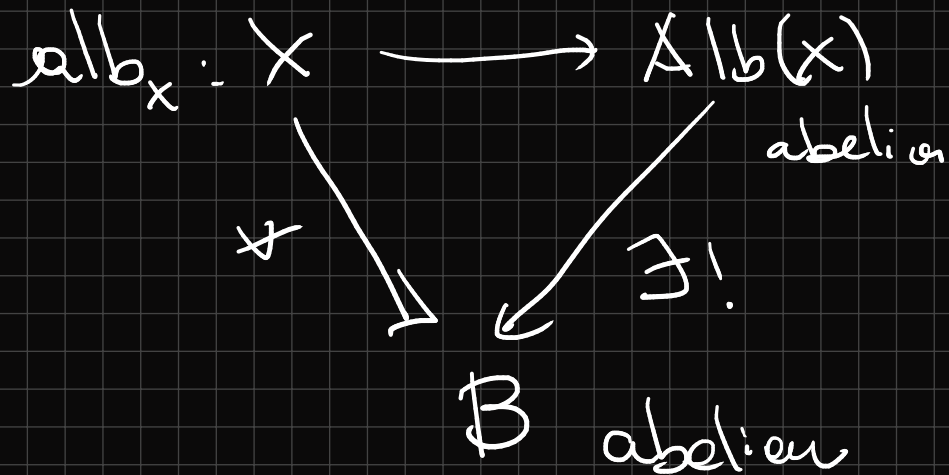
$$\overline{q}(G) = \dim G$$

$$\overline{P}_m(G) = 1 \quad \forall m$$

$$\overline{k}(G) = 0$$

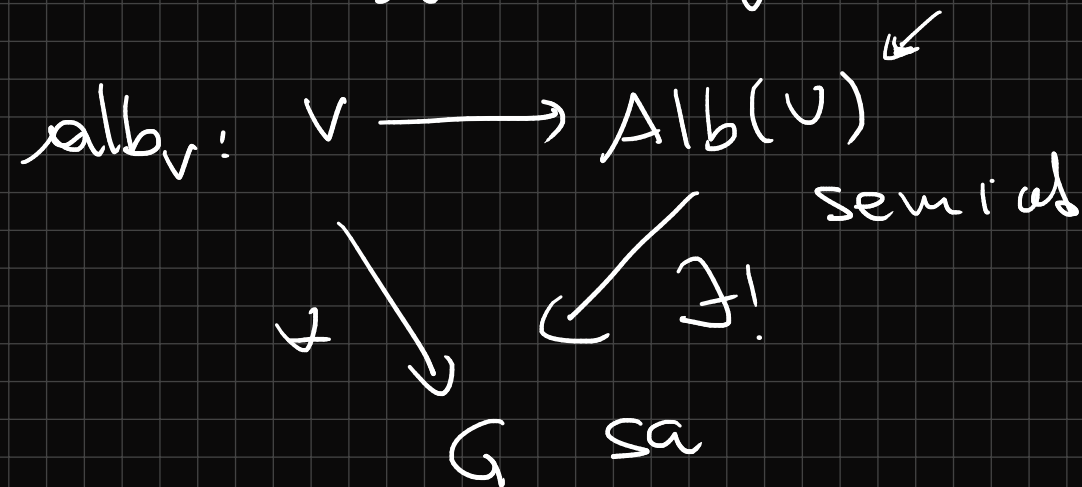
X smooth projective

$\exists (Alb(X), alb_X)$



V quasi projective

$\exists (Alb(V), alb_V)$



$$0 \rightarrow G_m^r \rightarrow Alb(V) \rightarrow Alb(X) \rightarrow 0$$

X opt
of V

Def: We say that V is of maximal
Albanese dimension if

$$\text{alb}_V: V \longrightarrow \text{Alb}(V)$$

$$\dim V = \dim \text{alb}_V(V)$$

Results:

• Iitake 1980, Kawamata 1981 (+ Fujino-Hendos Lopes - Pardini - ²⁰²⁴!)

We had a characterization of semi-abelian varieties up to WWPB equivalence

$$\bullet \quad \overline{K}(V) = 0 \quad \overline{q}(V) = \dim V$$

Thm1: V smooth & projective of m Ad

(Baudin with $\overline{P}_1(V) = \overline{P}_2(V) = 1$

!)

$\Rightarrow V$ is WWPB equivalent

to a semi-abelian variety

Thm 2 : V smooth quasiprojective.

$$\overline{P}_1(V) = \overline{P}_2(V) = 1 \quad \overline{q}(V) = \dim V$$

we prove in Ad1
under this extra
assumption.

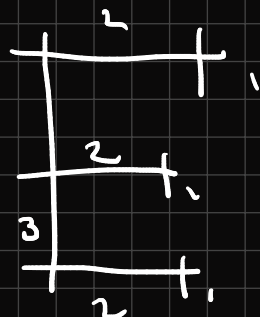
$$\boxed{q(x) \geq \overline{q}(V) - 1}$$

$\Rightarrow V$ is WWPB equivalent to
a semi-abelian

Example (Hendres Lopez - Pardini - ")

S elliptic K3 surface with 3 singular
fibers of type I_n^*

($\exists$)



F_i support
of the i -th
singular fiber.

$$V = X \setminus \{\#_1, \#_2, \#_3\}$$

V smooth quasi projective surface

$$\overline{\Phi}_1(v) = \overline{\Phi}_2(v) = 1$$

$$\overline{q}(v) = 2$$

but the alb_v is
not dominant

Difficult to show maximal abelian
dimension starting from $\Phi_n(v)$ especially
when $q(x)$ is low

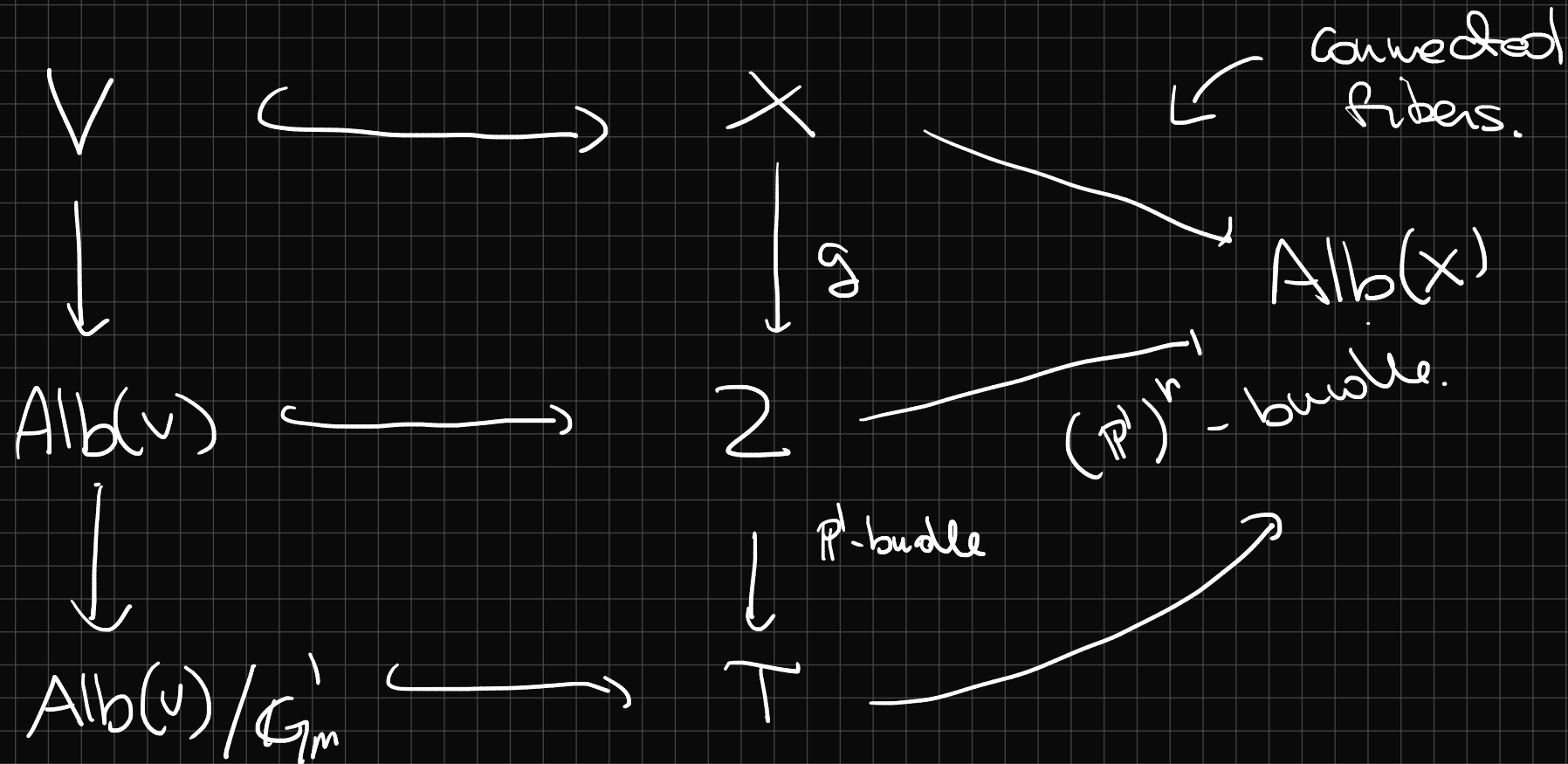
Conjecture: There should be $m_0(\underbrace{\bar{q}(v) - q(x)})$
such that

$$\overline{P_1(v)} = \overline{P_{m_0}(v)} = 1$$

$$\bar{q}(v) = \dim(v)$$

$\Rightarrow V$ is wwpB equivalent
to a semi abelian.

Cor V affine + mAd with $P_1(v) - P_2(v) = 1$
 $\Rightarrow V \cong G_m^r$



WANT

- g birational
- g contract the components of D which do not map in Δ

g is generically finite.

$$K_X + D \sim g^*(K_Z + \Delta) + \overline{R}_g$$

L , litaka
log ramificati
divisor.

• $\Gamma \subseteq \overline{R}_g$ which does not map to Δ
 $\Rightarrow \Gamma$ does not dominate $\text{Alb}(X)$

Baby case

$g(X) = 0$ $\text{Alb}(X) = \{pt\}$ $\Rightarrow \Gamma$
 does not exist.

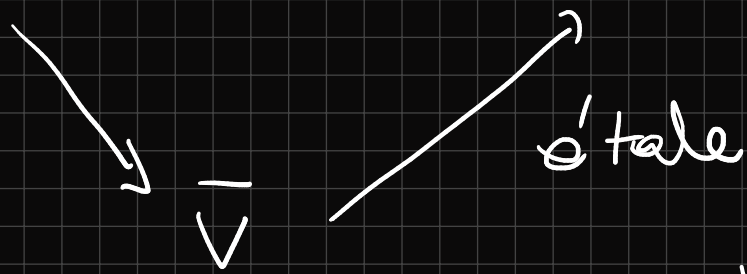
$$g: X \longrightarrow Z$$

$$\searrow \overline{X} \xrightarrow{\overline{g}}$$

stein factorizati

$\overline{g}$ is étale over $\text{Alb}(Y)$

$$\text{alb}_V: V \longrightarrow \text{Alb}(V)$$



$\Rightarrow$ semi abélien
 $\Rightarrow$ universal
 propriété

$$V \simeq \text{Alb}(V)$$

$$\underline{g(x)=1}$$

$$\Gamma \backslash \mathbb{H}^n$$

$$\downarrow$$

$$Z \xrightarrow[\varphi]{} E$$

the assumptions on the log plurigenere

$$pg(\pi) = pg(\text{theta divisor})$$

Topology argument

$$\pi_1(E) = \pi_1(E, p)$$

Logic

$$g: \overline{X} \rightarrow Z \text{ étale over } \text{Alb}(Y)$$

